Euclidean and Non-Euclidean Geometry - Week 3

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The Postulates

P1 (The Set Postulate) Every line is a set of points, and there is a set of all points called the plane.

With only this postulate, we can't say anything as there may be no points in the plane at all! So we need the following to get started:

P2 (The Existence Postulate) There exists at least three distinct non-collinear points.

Now have some points, but we may have no lines. The following postulate ensures that we have lines as well:

P3 (The Unique Line Postulate) Given any two distinct points, there is a unique line that contains both of them.

Theorem

Two distinct lines intersect at either no points or one point.

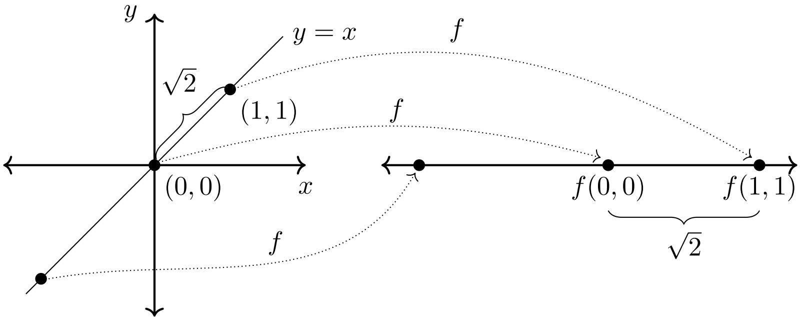
The Postulates

P4 (The Distance Postulate) For every pair of points A and B, the distance from A to B is a non-negative real number determined by A and B.

P5 (The Ruler Postulate) For every line ℓ, there is a one-to-one correspondence between points on the line and the real numbers such that if A and B lie on the line, then the distance between A and B is the distance between the corresponding real numbers.

P5 essentially says that we can measure the distances in P4 (in particular, they aren't assigned randomly).

The Postulates



The Postulates

Theorem

There are infinitely many points on every line, and there are infinitely many lines in the plane.

P6 (The Plane Separation Postulate) For any line ℓ, the set of all points not on ℓ is the union of two disjoint subsets called the sides of ℓ. If A and B are distinct points on ℓ then A and B are on the same side of ℓ if and only if the line segment joining A and B does not intersect ℓ.

P7 (The Angle Measure Postulate) For every angle $\angle ab$, the measure of $\angle ab$ is a real number between 0 and 180 determined by $\angle ab$.

P8 (The Protractor Postulate) *This postulate is hard to state, but it essentially does for angles what the Ruler Postulate does for lengths.*

The Postulates

A **triangle** *is the union of three line segments formed by three non-collinear points. Note, we can discuss line segments thanks to P5.*

P9 (The SAS Postulate) If two triangles have a pair of sides of the same length, with the corresponding angle having the same measure, then the two triangles are congruent (all three sides and angles are the same).

Theorem

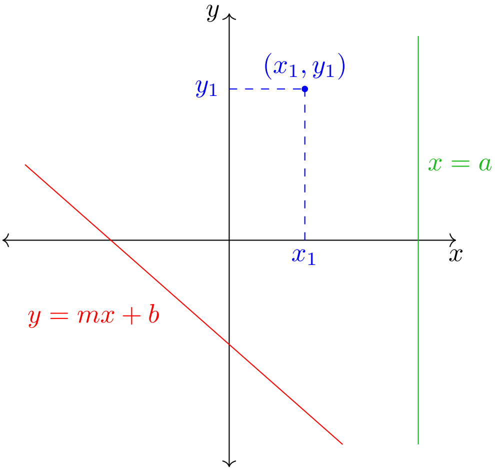
For each line ℓ and each point P that does not lie of ℓ, there is a line that contains P and is parallel to ℓ.

What this result does not tell us is how many such lines there are. The following postulate asserts there is only one:

P10 (The Parallel Postulate) For each line ℓ and each point P that does not lie on ℓ, there is a unique line that contains P and is parallel to ℓ.

Euclidean Geometry

The Cartesian plane with points (x_1, y_1) *and lines of the form* $y = mx + b$ *or* $x = a$ *satisfy P1 - P10 where distance is measured in the usual way (by Pythagoras' Theorem) and so are angles.*



Euclidean Geometry

Theorem

Nothing else satisfies P1 - P10. That is, if P1 - P10 hold, you get (something isomorphic to) the Cartesian plane with points (x_1, y_1), lines of the form $y = mx + b$ or $x = a$, where distance is measured in the usual way (by Pythagoras' Theorem), and so are angles.

This tells us that postulates P1 - P10 are enough to completely determine the geometry we're familiar with. Do we need all 10 though?

Last time, we saw that Taxicab Geometry (which replaces the usual distance with taxicab distance) satisfies P1 - P8, but not P9. So to get Euclidean geometry, we at least need P1 - P9. Do we need P10?

Revisiting The Parallel Postulate

Theorem

For each line ℓ and each point P that does not lie of ℓ, there is a line that contains P and is parallel to ℓ.

What this result does not tell us is how many such lines there are. The following postulate asserts there is only one:

P10 (The Parallel Postulate) For each line ℓ and each point P that does not lie on ℓ, there is a unique line that contains P and is parallel to ℓ.

Is P10 really necessary? Can we prove that the parallel line through P is unique just using P1 through P9?

In an effort to answer this question, people looked for alternative statements which were equivalent to P10 and hoped to prove them instead.

Revisiting The Parallel Postulate

Theorem

If P1 though P9 are satisfied, then the following are equivalent:

- The Parallel Postulate (P10).
- If ℓ_1, ℓ_2, ℓ_3 are lines with ℓ_1 parallel to ℓ_2, and ℓ_2 parallel to ℓ_3, then ℓ_1 is parallel to ℓ_3.
- If ℓ_1 and ℓ_2 are parallel, and ℓ_3 intersects ℓ_1, then ℓ_3 intersects ℓ_2.
- The sum of angles in a triangle is $180°$.
- Pythagoras' Theorem.
- There exists a rectangle.
- Given three non-collinear points, there is a circle that passes through them.

This shows that if P1 - P9 are satisfied but P10 is not satisfied, then it is a form of geometry very different from what we are used to.

Revisiting The Parallel Postulate

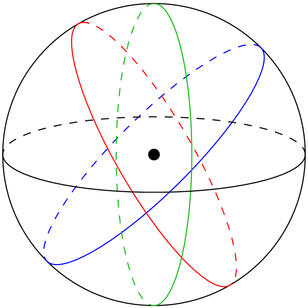
Question. (300 BCE) If P1 - P9 are satisfied, is P10 necessarily satisfied? That is, can one use P1 through P9 to prove that for each line ℓ and each point P that does not lie on ℓ, there is a unique line that contains P and is parallel to ℓ?

Answer. (Over 2000 years later) No! That is, you cannot prove the statement of P10 only using P1 through P9.

How does one show this? Give an example of a plane satisfying P1 - P9 which does not satisfy P10 (or one of the equivalent statements).

Spherical Geometry

Consider a sphere – this will be our plane. The points will be the points of the sphere. The lines are the great circles - circles on the sphere whose centre is the same as the centre of the sphere.



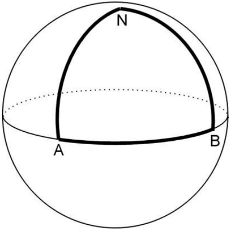
Spherical Geometry

Riddle: Where in the world can you travel 10km south, 10km west, and then 10km north and end up back where you started?

One possibility is the north pole. But there are infinitely many others too.

Spherical Geometry

Consider the following triangle. Start at the north pole and travel along a great circle to the equator. Then travel along the equator a quarter of the way. Then travel along a great circle back to the north pole.



All three angles are 90°. *So the triangle has angle sum* 270°.

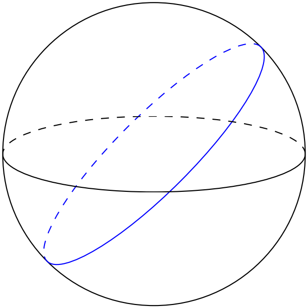
Spherical Geometry doesn't satisfy P10!

Spherical Geometry

P1 (The Set Postulate) Every line is a set of points, and there is a set of all points called the plane. ✓

P2 (The Existence Postulate) There exists at least three distinct non-collinear points. ✓

P3 (The Unique Line Postulate) Given any two distinct points, there is a unique line that contains both of them. ✗



Breakthrough

In the late 1820's and early 1830's, Nikolai Lobachevsky and János Bolyai independently published work on **non-Euclidean geometry** *(a geometry where P1 - P9 is satisfied, but P10 is not). Carl Friedrich Gauss also claimed to have made similar observations in unpublished work*.

It wasn't until 1868 that Eugenio Beltrami demonstrated that it was possible for P1 - P9 to be satisfied and for P10 to be false, i.e. that non-Euclidean geometry actually exists!

The geometry they all studied is now known as **hyperbolic geometry***.*

Hyperbolic Geometry

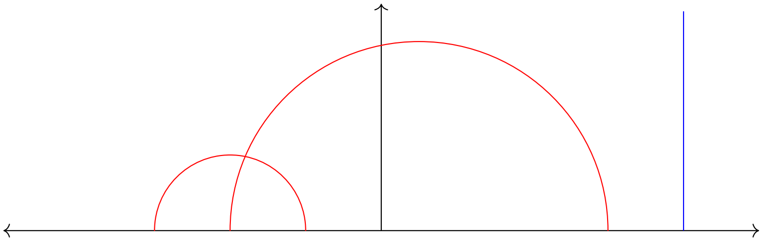
Just as when describing spherical geometry, in order to discuss hyperbolic geometry, we have to declare:

- what the plane is,
- what the points are,
- what the lines are,
- how to measure distances, and
- how to measure angles.

There are many ways to do this. They lead to different (isomorphic) models of hyperbolic geometry.

The Upper Half Plane Model

The plane will be the upper half of the Cartesian plane. That is, our points will be points (x, y) with $y > 0$. The lines will either be vertical lines in the upper half plane, or semicircles with centre on the x-axis.



The Upper Half Plane Model

Denote the distance between (x_1, y_1) *and* (x_2, y_2) *in the upper half plane model by* $\mathrm{dist}((x_1, y_1), (x_2, y_2))$.

Denote the Euclidean distance between them by $\|(x_2, y_2) - (x_1, y_1)\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The two are related as follows:

$$
\begin{aligned}
&\mathrm{dist}((x_1, y_1), (x_2, y_2)) \\
={}& 2\ln\left(\frac{\|(x_2, y_2) - (x_1, y_1)\| + \|(x_2, y_2) - (x_1, -y_1)\|}{2\sqrt{y_1 y_2}}\right) \\
={}& 2\ln\left(\frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}}{2\sqrt{y_1 y_2}}\right)
\end{aligned}
$$

The Upper Half Plane Model

$$\frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}}{2\sqrt{y_1 y_2}}$$

If y_1 and y_2 are small, then this quotient is large. If y_1 and y_2 are large, the quotient is large.

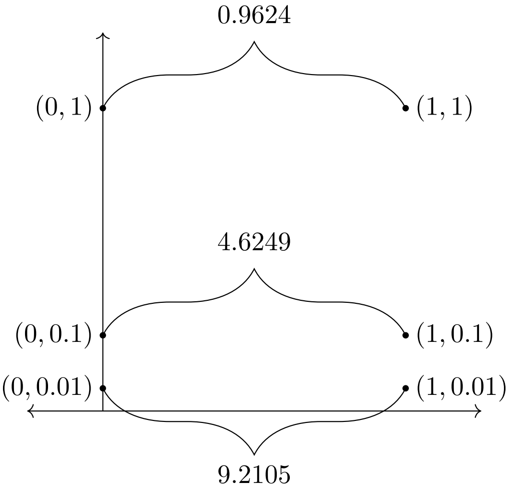
Example:

$$\operatorname{dist}((0,1),(1,1)) = \ln\left(\frac{1+\sqrt{5}}{2}\right) \approx 0.9624$$

$$\operatorname{dist}((0,0.1),(1,0.1)) = \ln\left(\frac{1+\sqrt{1.04}}{2}\right) \approx 4.6249$$

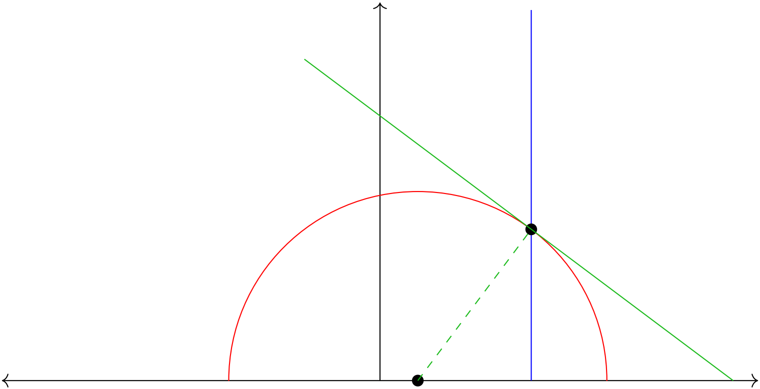
$$\operatorname{dist}((0,0.01),(1,0.01)) = \ln\left(\frac{1+\sqrt{1.0004}}{2}\right) \approx 9.2105$$

The Upper Half Plane Model



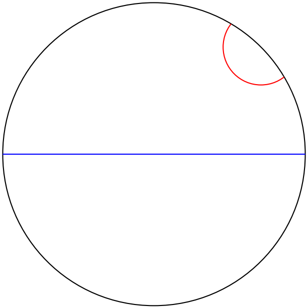
The Upper Half Plane Model

How to measure angles?



The Unit Disc Model

Another model of hyperbolic geometry is one for which the plane is the open unit disc. That is, the set of points (x, y) *with* $x^2 + y^2 < 1$. *The lines are either diameters of the circle, or semicircles meeting the boundary at right angles.*



The Unit Disc Model

The distance between (x_1, y_1) *and* (x_2, y_2) *in the unit disc model is:*

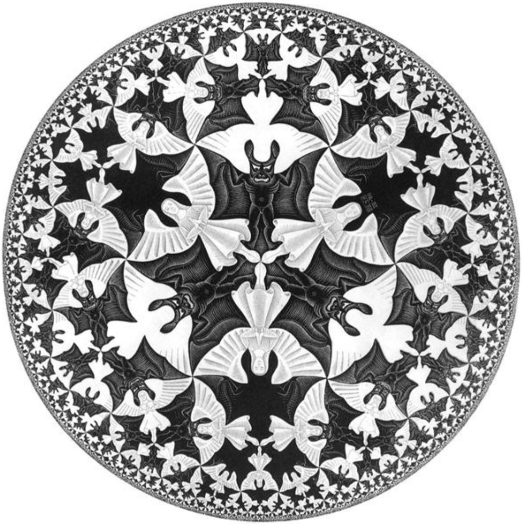
$$\ln\left(\frac{\sqrt{(1-x_1x_2-y_1y_2)^2+(x_2y_1-x_1y_2)^2}+\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}{\sqrt{(1-x_1x_2-y_1y_2)^2+(x_2y_1-x_1y_2)^2}-\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\right)$$

The expression is much simpler if you use complex numbers.

As you get closer the boundary, the distances increase (just as in the upper half plane model the distances increase as you get closer to the x*-axis).*

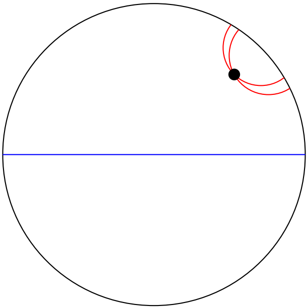
The Unit Disc Model

The Unit Disc Model



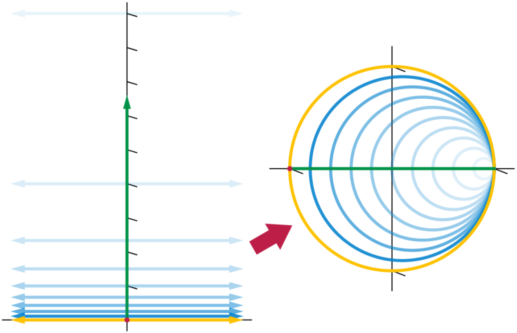
The Unit Disc Model

The unit disc model satisfies P1 - P9 but not P10.



How are the upper half plane model and the unit disc model related?

The Two Models



There are several other models, but they are all (essentially) equivalent.

Circles

A circle of centre (a, b) and radius r is the set of points (x, y) such that the distance from (x, y) to (a, b) is r. But this depends on which distance you're using.

$$\sqrt{(x-a)^2 + (y-b)^2} = r \Leftrightarrow (x-a)^2 + (y-b)^2 = r^2$$

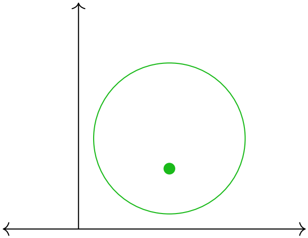
What do circles look like in the two models of hyperbolic geometry?

Circles

In the upper half plane model, a point (x, y) *belongs to the circle with centre* (a, b) *and radius* r *if*

$$2\ln\left(\frac{\sqrt{(x-a)^2+(y-b)^2}+\sqrt{(x-a)^2+(y+b)^2}}{2\sqrt{yb}}\right)=r$$

Miraculously, one can show that this is a regular circle in the upper half plane. However, the centre doesn't coincide with the usual centre in Euclidean geometry.

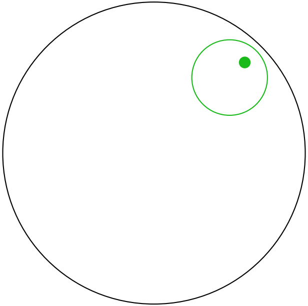


Circles

In the unit disc model, a point (x, y) *belongs to the circle with centre* (a, b) *and radius* r *if*

$$\ln\left(\frac{\sqrt{(1-ax-by)^2+(bx-ay)^2}+\sqrt{(x-a)^2+(y-b)^2}}{\sqrt{(1-ax-by)^2+(bx-ay)^2}-\sqrt{(x-a)^2+(y-b)^2}}\right)=r$$

By another miracle, one can show that this is a regular circle in the unit disc. However, the centre doesn't coincide with the usual centre in Euclidean geometry.



Circles

Theorem

In Euclidean geometry, the ratio between a circle's circumference and diameter is independent of the circle.

We denote this ratio by π*.*

What about in non-Euclidean geometry?

Theorem

In hyperbolic geometry, the ratio between a circle's circumference and diameter is NOT independent of the circle.

So there is no analogue of π *which holds for all circles. Depending on the circle, the ratio can be any number bigger than* π*!*

Circles

Exercise: What about Taxicab Geometry? What do the circles look like? Is there an analogue of π?